umns together with the last column of B'. Since the elements of the last column of Δ are linear functions of $x_{s+1}, \ldots, x_n$, Δ itself can be expressed as a linear function of the same variables, and their coefficients are r-rowed determinants involving the elements of the augmented matrix of the original system T. Thus

$$\Delta \equiv l_1 x_{s+1} + \ldots + l_t x_n + h = 0;$$

and l_1, being precisely the determinant of equation (2), does not vanish. We have now shown that all the values of $x_{s+1}, \ldots, x_n$ that satisfy the equations of the system T are connected by the same linear relation of the required form; therefore the theorem is proved, provided we can establish it for the case $t=1$.

But it is evident that the method we have used for passing from $t-1$ to t is also applicable to the case $t=1$, and so affords an independent proof of Professor Miller's theorem, besides completing the proof of this.

3. A number of corollaries are immediately deducible. For instance, if the unknowns $x_{s+1}, \ldots, x_n$ are unrestricted except by a single linear relation, or in other words, if in the solutions of the system T the values of these unknowns are precisely those that satisfy one linear relation of the form (1), then the rank of the matrix will be reduced exactly one unit by the omission of the coefficients of these unknowns, and conversely. On the other hand, if there exist i (and only i, say) independent and compatible linear relations connecting the t unknowns $x_{s+1}, \ldots, x_n$, then the rank r of the matrix will be reduced exactly i units, and conversely; evidently $i \leq t$ and $i \leq r$; if $i=t$, each of the t unknowns is restricted to a single value and the rank is reduced one unit by the omission of the coefficients of each of the t unknowns.

Moreover, if the relation (1) actually involves all of the t unknowns $x_{s+1}, \ldots, x_n$, that is, if $l_i \neq 0$ $(i=1, \ldots, t)$, and if there is no other linear relation between them, then the omission of the coefficients of all of these unknowns, and not merely of some of them, will reduce the rank of the matrix, and the amount of the reduction is one unit. The converse again holds.

Similarly, if the t unknowns are connected by i independent and compatible linear relations that actually involve all the unknowns, then the reduction in rank due to the omission of the coefficients of all of these t unknowns is equal to i, while the reduction due to the omission of the coefficients of a part of the unknowns is less than i, and conversely.

CLASSIFICATION.

4. Let us now consider the bearing of this theorem and its corollaries on the classification of systems of linear equations. Let T be a consistent

system of rank r in n unknowns. After a suitable permutation of the unknowns among themselves, the complete solution of T may be written in the form

$$S\begin{cases} x_1 = l_{1\,1}x_{r+1} + \ldots + l_{1,\,n-r}\,x_n + h_1 \\ \dot{x}_r = \dot{l}_{r1}\dot{x}_{r+1} + \ldots + \dot{l}_{r,\,n-r}\,\dot{x}_n + \dot{h}_r, \end{cases}$$

which means that all the solutions of T are obtainable by giving all possible values to $x_{r+1}, \ldots, x_n$ and for each set of values of these unknowns finding the corresponding values of the other unknowns $x_1, \ldots, x_r$ from the equations S.

Each of the equations S is of the form of equation (1), since the coefficients of $x_1, \ldots, x_r$ cannot vanish. But the remaining coefficients l_{ij} may some or all of them vanish, and if we classify the systems T with respect to the vanishing or non-vanishing of these coefficients, that is, by determining which of the unknowns $x_{r+1}, \ldots, x_n$ are actually involved in the various equations of S, the different classes so obtained (see § 5) will be distinguished from one another and characterized by the particular sets of unknowns, the omission of whose coefficients from the matrix of T will cause reductions in the rank of the matrix.

Since no two of the equations S involve the same unknowns, those on the left being different in every case, therefore the reduction in rank caused by the omission of the coefficients of (a) all, or (b) a part of the unknowns actually involved in any one of the equations of S is equal to (a) one unit or (b) zero units.

We define the symbol $R_{x_\alpha \ldots x_\lambda}$ as the number of units of reduction in rank that is caused by the omission of the coefficients of $x_\alpha, \ldots, x_\lambda$ from the matrix of T; and we understand in every case by an equation of the form $R_{x_\alpha \ldots x_\lambda} = j$, that the omission of the coefficients of a part of these unknowns will reduce the rank less than j units, except when $j=0$. As an illustration, let T be a system of rank 3 in five unknowns whose complete solution S is of the form

$$\begin{cases} x_1 = l_{1\,1}x_4 + l_{1\,2}x_5 + h_1 \\ x_2 = l_{2\,1}x_4 \qquad\quad + h_2 \\ x_3 = \qquad\qquad\quad h_3 \end{cases}$$

where $l_{1\,1}$, $l_{1\,2}$, $l_{2\,1}$ are all different from zero; then the class of systems to which T belongs will be characterized by the equations $R_{x_1\,x_4\,x_5} = R_{x_2\,x_4} = R_{x_5} = 1$; moreover,

$$R_{x_1\,x_4} = R_{x_2} = R_{x_4} = 0, \quad R_{x_1 x_2 x_4 x_5} = R_{x_2\,x_3\,x_4} = 2, \quad \text{and} \quad R_{x_1 x_2 x_3 x_4 x_5} = 3.$$

5. Now the number of ways in which the $r(n-r)$ coefficients l_{ij} in the equations S can be chosen with respect to their vanishing or not is $2^{r(n-r)}$. But these do not all give essentially distinct classes of systems T. For if two such choices differ only in the order in which the unknowns are arranged, that is, if one can be obtained from the other by permuting either the $n-r$ unknowns $x_{r+1}, \ldots, x_n$ on the right or the r unknowns $x_1, \ldots, x_r$, on the left, then they give rise to essentially the same class; otherwise to distinct classes.

Let $N_{n,\,r}$ denote the number of classes of systems of consistent linear equations of rank r in n unknowns. Then $N_{n,\,r}$ is equal to the total number of combinations of $r(n-r)$ things arranged in the form of a rectangle with r rows and $n-r$ columns, provided that two combinations be regarded as identical, if one can be obtained from the other by permuting the rows of this rectangle or the columns or both. Obviously, $N_{n,\,r}=N_{n,\,n-r}$.

Query. Is there a convenient formula, old or new, giving the value of $N_{n,\,r}$ in terms of n and r?

For a few of the smaller values of n and r it is not difficult to find the corresponding values of $N_{n,\,r}$, as follows:

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	7	4	1		
$n=5$	1	5	13	13	5	1	
$n=6$	1	6	22	32	22	6	1

Geometrical Interpretation.

6. If we interpret $x_1, \ldots, x_n$ as the Cartesian coordinates of a point in a space S_n of n dimensions,* then a system T of consistent linear equations of rank r in these variables will stand for a system of linear (flat) spaces S_{n-1} intersecting in a linear space S_{n-r}; and the different classes we have been considering will correspond to the different degrees of parallelism† that can exist between S_{n-r} and the coordinate axes, coordinate planes, etc.

On the other hand, if the equations of T are homogeneous in the n variables, in which case they are always consistent, then another geometrical interpretation is possible. Namely, let the variables be the homogeneous coordinates of a point in a space S_{n-1}, referred to a coordinate simplex.‡ T will now stand for a system of linear spaces S_{n-2} interecting in a linear space

*Cf. Schoute, *Mehrdimensionale Geometrie*, Vol. 1, pp. 29, 125, and 137.
†Schoute, l. c., p. 34.
‡Schoute, l. c., p. 142.

S_{n-r-1}, except when $r=n$; in the latter case the spaces have no point in common. The classification we have been considering will correspond to the different degrees of intersection, including coincidence and non-intersection, that exist between S_{n-r-1} and the vertices, edges, faces, etc., of the coordinate simplex.

ILLUSTRATIONS.

7. In order to illustrate the general theory, a few of the simpler special cases will now be mentioned.

First, we put $n=3$ and enumerate the different classes of consistent systems T of linear non-homogeneous equations in the three variables x, y, z for each possible value of the rank r. Geometrically, T is a system of planes having at least one point in common, and referred to Cartesian axes, retangular or oblique, in ordinary space of three dimensions. Let S_{3-r} be the locus common to these planes.

$r=3$. Every system T has one solution. The number of classes of systems is $N_{3,3}=1$. This one class is characterized by the equations $R_x=R_y=R_z=1$, or by the single equation $R_{xyz}=3$. The common locus S_0 of the planes is a finite point.

$r=2$. T has ∞^1 solutions. The number of classes is $N_{3,2}=3$. Call these classes (2_1), (2_2), and (2_3), respectively.

(2_1) $R_{xz}=R_{yz}=1$. Moreover, $R_{xy}=1$.

The common locus S_1 is a (finite) line not parallel* to a coordinate plane.

(2_2) $R_x=R_{yz}=1$. S_1 is a line parallel to the plane $x=0$, but not parallel to a coordinate axis.

(2_3) $R_x=R_y=1$ or $R_{xy}=2$. S_1 is a line parallel to the z-axis.

$r=1$. T has ∞^2 solutions. The number of classes is $N_{3,1}=3$.

(1_1) $R_{xyz}=1$. S_2 is a plane not parallel to a coordinate axis.

(1_2) $R_{xy}=1$. S_2 is a plane parallel to the z-axis, but not parallel to a coordinate plane.

(1_3) $R_x=1$. S_2 is a plane parallel to $x=0$.

8. As a second illustration we take homogeneous equations and put $n=4$. Let x, y, z, w, be the homogeneous coordinates of a point in three-dimensional space referred to a coordinate tetrahedron. T is again a system of planes and S_{3-r} is their common locus, except when $r=4$.

$r=4$. One solution, $x=y=z=w=0$. $N_{4,4}=1$.

$R_x=R_y=R_z=R_w=1$. No common locus.

$r=3$. ∞^1 solutions. $N_{4,3}=4$.

(3_1) $R_{xw}=R_{yw}=R_{zw}=1$. S_0 is a (finite or infinite) point not lying in a coordinate plane.

*It is to be understood here that two parallel lines or planes may, in particular, be coincident lines or planes, and that a line parallel to a plane may be a line lying in the plane.

(3_2) $R_x=R_{yw}=R_{zw}=1$. S_0 is a point lying in the plane $x=0$, but not in a coordinate axis.

(3_3) $R_x=R_y=R_{zw}=1$. S_0 is a point in the line $x=y=0$, but not coinciding with a coordinate vertex.

(3_4) $R_x=R_y=R_z=1$. S_0 is the point $x=y=z=0$.

$r=2$. ∞^2 solutions. $N_{4,2}=7$.

(2_1) $R_{xzw}=R_{yzw}=1$. S_1 is a line not meeting a coordinate axis (either in a finite or an infinite point).

(2_2) $R_{xw}=R_{yzw}=1$. S_1 is a line meeting $x=w=0$, but not meeting any other coordinate axis.

(2_3) $R_{xw}=R_{yz}=1$. S_1 is a line meeting the two opposite coordinate axes $x=w=0$ and $y=z=0$, but not lying in a coordinate plane.

(2_4) $R_x=R_{yzw}=1$. S_1 is a line lying in the plane $x=0$, but not passing through a coordinate vertex.

(2_5) $R_{zz}=R_{yz}=1$. S_1 is a line passing through the point $x=y=z=0$, but not lying in a coordinate plane.

(2_6) $R_x=R_{yz}=1$. S_1 is a line passing through the point $x=y=z=0$, and lying in the plane $x=0$, but not coinciding with a coordinate axis.

(2_7) $R_x=R_y=1$. S_1 is the line $x=y=0$.

$r=1$. ∞^3 solutions. $N_{4,1}=4$.

(1_1) $R_{xyzw}=1$. S_2 is a plane not passing through a coordinate vertex.

(1_2) $R_{xyz}=1$. S_2 is a plane passing through the point $x=y=z=0$, but not containing a coordinate axis.

(1_3) $R_{xy}=1$. S_2 is a plane containing the line $x=y=0$, but not coinciding with a coordinate plane.

(1_4) $R_x=1$. S_2 is the plane $x=0$.

9. It may be worth while to add the four-dimensional interpretation of the seven classes (2_1), ..., (2_7) just given, for which $n=4$ and $r=2$, under the supposition that the equations of T are again non-homogeneous.

(2_1) S_2 is a plane having neither complete nor partial parallelism to any coordinate plane.

(2_2) S_2 is a plane half parallel to $x=w=0$, but neither completely parallel nor half parallel to any other coordinate plane.

(2_3) S_2 is a plane half parallel to the two opposite coordinate planes $x=w=0$ and $y=z=0$, but not (completely) parallel to a coordinate space S_3.

(2_4) S_2 is a plane parallel to the space $x=0$, but not parallel to a coordinate axis.

(2_5) S_2 is a plane parallel to the line $x=y=z=0$, but not parallel to a coordinate space.

(2_6) S_2 is a plane parallel both to the line $x=y=z=0$ and to the space $x=0$, but not completely parallel to a coordinate plane.

(2_7) S_2 is a plane completely parallel to $x=y=0$.

THE FOUNDER OF GROUP THEORY.

By G. A. MILLER, University of Illinois.

In the very useful *Encylcopédie des Sciences Mathematiques*, tome I, volume 1 (1909), page 432, it is stated that Cauchy may be considered as the founder of the theory of substitution groups. On the contrary, Easton affirms in his *Construtcive development of group theory*, 1902, page 43, "The proper founder of group theory is Evariste Galois." Again, Pierpont has stated in the *Bulletin of the American Mathematical Society*, vol. I (1895), page 196, "in the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope by presenting a few examples of his methods to show the importance of considering him from this point of view." A still different view is expressed by Burkhardt in his important article entitled "Die Anfaenge der Gruppen theorie und Paolo Ruffini" published in the *Abhandlungen zur Geschichte der Mathematik*, 1892, pages 119-159. In concluding this paper he says: "It will probably not be possible to determine whether the essentials of the work of his friend Abbati, which includes the first complete proof of two fundamental theorems, are due to the inspiration of Ruffini, or whether we should regard this almost forgotten man alongside with Ruffini as one of the founders of group theory."

From the preceding paragraph it is clear that within the last twenty years four different names (Cauchy, Galois, Lagrange, and Ruffini) have been given by good authorities as those of founders of the theory of groups. Among others for whom this honor has been claimed Abel is perhaps the best known. In his well known *Synopsis der hoeheren Mathematik*, 1891, page 287, Hagen says "The theory of substitution groups was founded by Abel and Cauchy, and was developed to a certain extent principally by Galois and Jordan, especially with a view to the solution of algebraic equations." On the other hand, in Maillet's, Paris, thesis, 1892, we read "The founders of the theory (of substitution groups) at least in its present form are Galois and Cauchy." These quotations may suffice to show how difficult it is to determine the founder of a large subject even of comparatively recent origin. As regards the older subjects, for instance, analytic geometry and calculus, the difficulty is generally much greater.

We are inclined to attribute the honor of starting a given big theory to an individual just as we are prone to ascribe fundamental theorems to particular men, who frequently have added only a small element to the development of the theorem. Hence the statement that a given individual founded a big theory should not generally be taken very seriously. It adds, however, a pleasant human flavor and awakens in us a noble sense of admiration and appreciation. It is also of value in giving a historical setting and brings into play a sense of the dynamic forces which have contributed to its

development instead of presenting to us a cold static scene. Observations become more inspiring when they are permeated with a sense of development.

The earliest of the names proposed as founder of the group theory is that of Lagrange, and his claim is based upon his admirable paper on the algebraic solutions of equations, published in the Berlin *Nouveaux Mémoires*, 1770 and 1771. If we accept Lagrange as the founder of group theory this subject is now about one hundred forty years old. At any rate, all the later men for whom the honor of having founded group theory is claimed were inspired by the beautiful theorems of this memoir. Among these is the fundamental theorem that the order of a subgroup is a divisor of the order of the group, which is sometimes called Lagrange's theorem.* It is scarcely necessary to add that Lagrange did not state this theorem, article 104 of his memoir, in the form in which we give it now. His language seems however practically equivalent to the given statement and is more accurate than one would naturally infer from the incorrect statement in Pierpont's review in the *Bulletin of the American Mathematical Society*, volume I (1895), top of page 198.

From the given quotations it may be inferred that Cauchy is more commonly regarded as the founder of group theory than any of the others. The opening sentence of the preface of Burnside's work on this subject, "The theory of groups of finite order may be said to date from the time of Cauchy" is in accord with such an inference. We proceed to consider some reasons for this conclusion. In the first place, it is desirable to emphasize the fact that Cauchy's contributions to group theory may be conveniently divided into two parts which are separated by a period of about thirty years. The first of these consists almost entirely of two articles published in 1815 in volume 10 of the oldest extant mathematical periodical, *Journal de l'école polytechnique*, while the second part begins with volume 3 of his *exercises d'analyse et de physique mathematique*, 1844, and closes with the numerous articles which he published in the Paris *Comptes Rendus* during 1845 and 1846.

The first of these periods is subsequent to the works of Lagrange and Ruffini but it antecedes those of Abel and Galois, while the second period is subsequent also to the works of the latter. During this second period Cauchy made his most important as well as his most extensive contributions to our subject, and these are the contributions which appear to justify the claim that he is the founder of group theory. His contributions of 1815 are scarcely as meritorious as the earlier ones by Lagrange or Ruffini, and hence these would not justify the claim that he is the founder of this theory. The assumption that Cauchy is the founder of group theory therefore implies that this theory is less than seventy years old.

One of the fundamental theorems proved by Cauchy during the

*Cf. Pincherli, *Lezioni di Algebra Complementare*, 1909, p. 44.

second period of his activity along this line is, Every group whose order is divisible by a given prime (p) must have a subgroup of order p. This is sometimes called Cauchy's theorem, and it constitutes the most difficult element in the development of Sylow's theorem. After Lagrange proved that the order of a subgroup is a divisor of the order of the group, and Ruffini established the theorem that it is not always possible to find a subgroup whose order is a given divisor of the order of the group (Ruffini's theorem), it was of great interest to establish the fact that a subgroup exists for every possible prime divisor. These three theorems, associated with the names of Lagrange, Ruffini, and Cauchy, respectively, constitute the most indispensable elements for the further development of the subject.

While Cauchy's theorem is comparable with those of Lagrange and Ruffini as regards fundamental importance for the further development of group theory, its proof demands a decidedly deeper insight into the nature and structure of a group. In this respect a fundamental theorem proved by Cauchy during his first period of group-theoretic activity is more nearly comparable with those of Lagrange and Ruffini. This theorem established the fact that the symmetric group of degree n cannot involve a subgroup whose index lies between 2 and p, where p is the largest prime which divides n. This theorem is a second extension of one due to Ruffini (1799), the first extension having been made by Abbati in 1803. It was extended still further by Bertrand, Serret, and others, and is of great importance in the study of the number of different values which a function may assume when its variables are permuted in every possible way.

Having considered the most important theorems contributed by Lagrange, Ruffini, and Cauchy towards the development of group theory it is of interest to inquire into the contributions of Galois, whose name has also been advanced as that of an individual founder of this subject. Perhaps his most important direct contribution is the introduction of the concept of modulus into group theory, a concept which Gauss had introduced into number theory about thirty years earlier (1801). In group theory, the modulus is generally known as invariant subgroup, although Jordan used the term modulus, at least indirectly, for this concept as early as 1873, in the *Bulletin of the French Mathematical Society*, volume I, page 46. In group theory this important concept is known by the following names, in chronological order: proper divisor, modulus, distinguished subgroup, invariant subgroup, monotypic subgroup, self-conjugate subgroup, normal divisor, and autojugan divisor. The multitude of names is perhaps partly due to the many different ways of approach to this concept.

Although Cauchy has contributed a number of important theorems to the development of group theory his claims as founder of this theory are more strongly supported by the fact that during the second period of his activity along this line he made the first systematic study of the theory of substitution groups, under the name of systems of conjugate substitutions.

While his developments are often prolix and involve some inaccuracies, they have placed a considerable part of the theory of substitutions into an easily accessible form and have been a source of inspiration for many of his successors. As an instance of an inaccuracy we may cite his statement (on page 443, volume 9, of the first series of his works) that a primitive group whose degree is a prime number increased by one cannot be simply transitive. It is very easily seen that the symmetric group of degree 9 can be represented as a simply transitive primitive group of degree $84=83+1$. His enumeration of the possible orders of groups of degree 6, on page 493 of the same volume, is also far from correct; but this should perhaps not surprise us in view of the large number of errors in the published enumerations of possible substitution groups.

The preceding considerations neither prove nor disprove the justice of the claim that Cauchy is the founder of group theory even if they tend to support this view. It has been our aim to exhibit a few of the elements involved in such a question, and especially to point out that many efficient workers are needed for the development of a great subject. Mathematical subjects gain in attractiveness if we can associate with them an intelligent insight into their growth and a due appreciation of the costly heritage involved in their fundamental theorems. To this end it is desirable to associate one or more founders with each of the modern subjects.

PERFECT NUMBERS.

By T. M. PUTNAM, University of California.

The theory of numbers, probably more than any other branch of mathematics, offers problems that are very easy to state and formulate completely, but extremely difficult to solve. There are many that have baffled even trained workers in this field, who have been obliged to content themselves in many cases with but partial resolutions of the questions. These very often appear as isolated, artificial problems whose solution would apparently add very little to the main body of theory. But sometimes there is an historical interest attached, which coupled with an alluring simplicity of formulation attracts investigators toward it. There is always the possibility, too, that the pursuit of solutions of even these elusive problems may lead to the discovery of mathematical relations, or processes that are new and of much more general application than to the immediate problem to be solved.

Some such justification may be necessary for research concerning the existence or relations of perfect numbers. Indeed, Fermat was led by this problem to some of his most important theorems. It is moreover a problem of much historic interest.

A perfect number is defined to be one that is equal to the sum of all its divisors exclusive of itself. If one denotes such a number by m, and the sum of all its factors including itself by $\sigma(m)$, then the defining relation is

$$\sigma(m)=2m.$$

If $m=a^\alpha b^\beta \dots l^\nu$ where $a, b, \dots, l$ are distinct primes, then

$$\sigma(m)=(1+a+a^2+\dots+a^\alpha)(1+b+b^2+\dots+b^\beta)\dots(1+l+l^2+\dots+l^\nu).$$

Hence the defining condition becomes

$$2\,a^\alpha b^\beta \dots l^\nu = (1+a+\dots+a^\alpha) \dots (1+l+\dots+l^\nu).$$

Euclid showed that all numbers of the form $2^{n-1}(2^n-1)$, where 2^n-1 is a prime, are perfect numbers; but it was Euler who first showed that all even perfect numbers are necessarily of this form. These theorems are easily verified by the above defining condition.

The smallest perfect numbers given by Euclid's formula are 6 and 8, but it has so far produced only nine such numbers, owing to the difficulty of determining whether or not 2^n-1 is a prime, when n is large. The known cases are for the values $n=2, 3, 5, 7, 13, 17, 19, 31,$ and 61. It is not known whether or not, this formula contains an infinite number of perfect numbers.

The theory of *even* perfect numbers is therefore fairly complete. But no *odd* perfect number has been found, nor has their existence either been proved or disproved. It is possible, however, to set up certain restrictive theorems concerning them, assuming that they do exist, which so hedge them about that the chances of any surviving are at least extremely small.

Some of the more important of these theorems will be stated below, and one new one (v. 7) will be given with proof and some applications.

1. If an odd perfect number exists it has the form p^kA^2, where p and k are both of the form $4n+1$, and p is a prime not dividing A. (Lucas, *Theorie des Nombres*, p. 425).

2. From (1) follows that there are no perfect numbers of the form $4h+3$.

3. Bourlet gives the following (*Nouv. Ann.*, 1896, p. 297): If the divisors of a perfect number be denoted by d_i then $\Sigma\frac{1}{d_i}=2$.

4. It follows from (3) that no divisor of a perfect number can be perfect and there must be more divisors than the least prime.

5. There are no perfect numbers of the forms $p^k a^2b^2\dots l^2$ and $p^ka^4b^4\dots l^4$, where $p, a, b, \dots, l$ are primes.

6. $\frac{m}{\sigma(m)} > (1-\frac{1}{a})(1-\frac{1}{b})\ldots(1-\frac{1}{i}) < 2$ (Bourlet, l. c.) From this follows that $\phi(m) < \frac{1}{2}m$, *i. e.*, less than half the numbers smaller than m are prime to it. This theorem and the next one derived from it are useful in ruling out certain types of odd numbers from consideration.

7. If we denote by r the number of distinct primes in a number m, and suppose that all these primes are greater than $\frac{r}{\log 2}+1$, the number m cannot be perfect.

To prove this take the relation given in (6):

$$\frac{m}{\sigma(m)} > \frac{1}{1+\frac{1}{a-1}} \cdot \frac{1}{1+\frac{1}{b-1}} \cdot \ldots \cdot \frac{1}{1+\frac{1}{i-1}}.$$

But by hypothesis each prime is greater than

$$\frac{r}{\log 2}+1, \text{ hence } \frac{1}{1+\frac{1}{a-1}} > \frac{1}{1+\frac{\log 2}{r}}, \text{ etc.}$$

Therefore, $\frac{m}{\sigma(m)} > \left(\frac{1}{1+\frac{\log 2}{r}}\right)^r > \left(\frac{1}{1+\frac{\log 2}{n}}\right)^n \quad (n>r).$

Hence, $\frac{m}{\sigma(m)} > \lim_{n=\infty} \left(\frac{1}{1+\frac{\log 2}{n}}\right)^n$, that is, $\frac{m}{\sigma(m)} > \frac{1}{e^{\log 2}}$, or $2m > \sigma(m)$.

Hence m cannot be perfect.

Since $\frac{1}{\log 2} = 1.45+$ one can state the theorem for working purposes in the form: A number with r distinct prime factors all of which are greater than $\frac{3}{2}r+1$ cannot be perfect.

Applications of (7).

a) There are no odd perfect numbers of the form a^α. Here $j=1$, $\frac{3}{2}j+1=\frac{5}{2}$, and since 3 is the smallest possible prime, the proof follows at once.

b) There are no odd perfect numbers with only two distinct prime factors. Here $\frac{3}{2}j+1=4$. The only possible type is then $3^\alpha \cdot p^\beta$. But

$\frac{m}{\sigma(m)} > (1-\frac{1}{3})(1-\frac{1}{p})$, or $\frac{m}{\sigma(m)} > \frac{2}{3}(1-\frac{1}{p})$. But $\frac{m}{\sigma(m)} = \frac{1}{2}$, if m is perfect. Hence,

$$\tfrac{1}{2} > \tfrac{2}{3}(1-\frac{1}{p}), \text{ or } p<4.$$

Hence there are no possible solutions.

c) If an odd perfect number could exist with three distinct primes, then one of these would have to be 3 or 5; for, $\frac{3}{2}j+1=5\frac{1}{2}$. But if we suppose that the form is $m=3^\alpha \cdot p^\beta \cdot q^\alpha$, then since

$$(1-\tfrac{1}{3})(1-\frac{1}{p})(1-\frac{1}{q})<\tfrac{1}{2}, \quad (1-\frac{1}{p})(1-\frac{1}{q})<\tfrac{3}{4}.$$

If $p=7$ and $q=11$, then $(1-\frac{1}{p})(1-\frac{1}{q})>\frac{3}{4}$, and larger values of p and q would also be impossible in this inequality. It follows therefore that one of them must be equal to 5. The form is then $m=3^\alpha \cdot 5^\beta \cdot p^\gamma \ldots$ The same form results if we suppose 5 present instead of 3, initially.

Since $1-\frac{1}{p}<\frac{15}{16}$ it follows that $p<16$.

The possible cases are then for $p=7$, 11, 13. Sylvester has shown that none of these combinations of three primes can lead to perfect numbers (*Comptes Rendus*, 1888). Hence no odd perfect number exists with less than four primes. Bourlet, investigating possible perfect numbers with four primes shows that none can exist less than 2,197,845.

This last theorem shows that if the prime factors of a number are all sufficiently large there is no possibility of it being perfect, no matter what the exponents may be. On the other hand, the third theorem expresses a relation involving all the divisors which implicitly involves the exponents of the prime factors of the number in such a way that they cannot exceed certain upper limits.

PROBLEM. Show that if an odd perfect number exists, say $p^k A^2$, then A has at least one prime factor smaller than p.

NOTE.—It may be of interest to some of our readers to know that Professor Benjamin Peirce, in 1832, *Mathematical Diary*, page 267, showed that there is no perfect number of the form a^m, $a^m b^n$, $a^m b^n c^p$, a, b, c being prime numbers greater than unity. ED. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

337. **Proposed by I. M. CURTISS, Brooklyn, N. Y.**

Three regiments move north as follows: B is 20 miles east of A; C is 20 miles south of B, and each marches 20 miles between the hours of 5 a. m. and 3 p. m. A horseman with a message from C starts at 5 a. m. and rides north till he overtakes B, then sets a straight course for the point at which he calculates to overtake A, then sets a straight course for the next point at which he will again overtake B, then rides south to the point where he first overtook B, reaching that point at the same time as C, namely 3 p. m. What uniform rate of travel enabled the messenger to do this?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let x=time required to overtake B. He travels $20+2x$ miles. Hence $\frac{20+2x}{x}$=his rate. Let y=time to go from B to A. He travels $2\sqrt{(100+y^2)}$ miles. $\frac{2\sqrt{(100+y^2)}}{y}$=his rate. After reaching B a second time he has left $10-x-2y$ hours to go $2x+4y$ miles.

$\therefore \frac{2x+4y}{10-x-2y}$=his rate. But his rate is uniform. Hence we get

$$\frac{20+2x}{x}=\frac{2\sqrt{(100+y^2)}}{y}, \text{ or } 5x^2=5y^2+xy\ldots(1).$$

$$\frac{20+2x}{x}=\frac{2x+4y}{10-x-2y}, \text{ or } x^2+2xy=50-10y\ldots(2).$$

If $x=vy$ in (1), we get $5v^2-v=5$ or $v=1.10499$.
$\therefore x=1.10499y$. This in (2) gives $14.41996y^2+10y=50$ or $y=1.54737$.
$x=1.70983$. Rate$=20/x+2=13.69707$ miles an hour.

Also solved by V. M. Spunar. A. H. Holmes, and J. Scheffer.

338. **Proposed by R. D. CARMICHAEL, Princeton University.**

Prove that $\pi=3+\frac{1}{3}\cdot\frac{1}{1.2}-\frac{1}{5}\cdot\frac{1}{2.3}+\frac{1}{7}\cdot\frac{1}{3.4}-\frac{1}{9}\cdot\frac{1}{4.5}+\ldots$

Solution by S. LEFSEHETZ, East Pittsburg, Pa.

$$\text{Let } S(x)=\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n(n+1)(2n+1)}=\sum_1^\infty\left[\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right](-1)^{n+1}x^{2n}$$

$$=\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n}+\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n+1}+4\sum_1^\infty(-1)^2\frac{x^{2n}}{2n+1}$$

$$=L(1+x^2)-\frac{1}{x^2}L(1+x^2)+1+\frac{4}{x}(\tan^{-1}x-x).$$

$$\therefore S(+1)=4(\frac{\pi}{4}-1)+1=\frac{1}{3.1.2}-\frac{1}{5.2.3}+\ldots$$

$$\therefore \pi=3+\frac{1}{3}.\frac{1}{1.2}-\frac{1}{5}.\frac{1}{2.3}+\ldots$$

Also solved by G. B. M. Zerr, and V. M. Spunar.

339. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that if $a_1<2$ and $a_n=a_{n-1}^2-2$, $\frac{1}{a_1}+\frac{1}{a_1 a_2}+\frac{1}{a_1 a_2 a_3}+\ldots$ $=\frac{1}{2}[a_1-\sqrt{(a_1^2-4)}]$.

Solution by the PROPOSER.

Let the roots of the equation $x^2-a_1x+1=0$ be α and $1/\alpha$. Then

$$a_1=\alpha+\frac{1}{\alpha},$$

$$a_2=\alpha^2+\frac{1}{\alpha^2},$$

$$a_3=\alpha^4+\frac{1}{\alpha^4},$$

. . . . .

The series becomes

$$\frac{1}{\alpha+\frac{1}{\alpha}}+\frac{1}{\left(\alpha+\frac{1}{\alpha}\right)\left(\alpha^2+\frac{1}{\alpha^2}\right)}+\ldots+\left(\alpha-\frac{1}{\alpha}\right)\left[\frac{1}{\alpha^2-\frac{1}{\alpha^2}}+\frac{1}{\alpha^4-\frac{1}{\alpha^4}}+\frac{1}{\alpha^8-\frac{1}{\alpha^8}}\ldots\right],$$

Since $\frac{1}{\alpha^2-\frac{1}{\alpha^2}}=\frac{1}{2}\left(\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}\right)$,

each term of the series may be written as the difference of two fractions, *i. e.*

$$\frac{\alpha-\frac{1}{\alpha}}{2}\left[\left(\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}\right)+\left(\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}-\frac{\alpha^4+\frac{1}{\alpha^4}}{\alpha^4-\frac{1}{\alpha^4}}\right)+\ldots.\right]$$

$$= \frac{a - \frac{1}{a}}{2}\left[\frac{a + \frac{1}{a}}{a - \frac{1}{a}} - \frac{a^{2^n} + a^{-2^n}}{a^{2^n} - a^{-2^n}}\right].$$

When $a>1$, the limit of this sum is

$$\frac{a - \frac{1}{a}}{2}\left(\frac{a + \frac{1}{a}}{a - \frac{1}{a}} - 1\right) = \frac{1}{a}.$$

Therefore, the sum of the series is the smaller root of the equation $x^2 - a_1 x + 1 = 0$, viz., $\frac{a_1 - \sqrt{(a_1^2 - 4)}}{2}$.

NOTE.—This furnishes a very rapid method for finding square roots to a considerable number of decimals. Example.—Let $a=16$,

$$8 - 3\sqrt{7} = \frac{1}{16} + \frac{1}{16.254} + \frac{1}{16.254.645.64514} + \ldots$$

and three terms of the series give $\sqrt{7}$ to 18 decimals, four terms will give 37 decimals, etc.

Also solved by S. Lefschetz.

GEOMETRY.

365. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Given the coordinates of the four vertices of the tetrahedron, (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3); (x_4, y_4, z_4): find volume and express it by a determinant.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and I. W. SMITH, A. M., Assistant Professor of Mathematics, North Dakota Agricultural College.

Let Δ = area of triangular base BCD of the tetrahedron, p the perpendicular from the vertex A on the base BCD, V its volume, $(x-x_2)\cos\alpha + (y-y_2)\cos\beta + (z-z_2)\cos\gamma = 0$ the equation to the plane of BCD.

Then $\Delta\cos\gamma$ is the projection of the area BCD on the plane xy, and (x_2, y_2), (x_3, y_3), (x_4, y_4) are its angular points.

$$\therefore 2\triangle\cos\gamma = \begin{vmatrix} 1, & 1, & 1 \\ x_2, & x_3, & x_4 \\ y_2, & y_3, & y_4 \end{vmatrix} = \begin{vmatrix} x_3-x_2, & x_4-x_2 \\ y_3-y_2, & y_4-y_2 \end{vmatrix}.$$

Similarly,

$$2\triangle\cos\beta = \begin{vmatrix} z_3-z_2, & z_4-z_2 \\ x_3-x_2, & x_4-x_2 \end{vmatrix}, \quad 2\triangle\cos\alpha = \begin{vmatrix} y_3-y_2, & y_4-y_2 \\ z_3-z_2, & z_4-z_2 \end{vmatrix}.$$

Also, $-p=(x_1-x_2)\cos\alpha+(y_1-y_2)\cos\beta+(z_1-z_2)\cos\gamma.$

$$\therefore -6V=-2\triangle p=2\triangle\cos\alpha(x_1-x_2)+2\triangle\cos\beta(y_1-y_2)+2\triangle\cos\gamma(z_1-z_2)$$

$$=(x_1-x_2)\begin{vmatrix} y_3-y_2, & y_4-y_2 \\ z_3-z_2, & z_4-z_2 \end{vmatrix}+(y_1-y_2)\begin{vmatrix} z_3-z_2, & z_4-z_2 \\ x_3-x_2, & x_4-x_2 \end{vmatrix}$$

$$+(z_1-z_2)\begin{vmatrix} x_3-x_2, & x_4-x_2 \\ y_3-y_2, & y_4-y_2 \end{vmatrix}$$

$$=\begin{vmatrix} x_1-x_2, & x_3-x_2, & x_4-x_2 \\ y_1-y_2, & y_3-y_2, & y_4-y_2 \\ z_1-z_2, & z_3-z_2, & z_4-z_2 \end{vmatrix} = -\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1-x_2, & 0, & x_3-x_2, & x_4-x_2 \\ y_1-y_2, & 0, & y_3-y_2, & y_4-y_2 \\ z_1-z_2, & 0, & z_3-z_2, & z_4-z_2 \end{vmatrix}$$

Multiply the first row of this last determinant by x_2, y_2, z_2 and add to the second, third, fourth, respectively, we get

$$V=\tfrac{1}{6}\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}$$

Also solved by S. G. Barton, and V. M. Spunar.

366. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester, N. H.

Construct a triangle, having given the base, vertical angle, and difference of altitude and difference of other two sides.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let A be the given vertical angle, a the given side, and d=given difference. Then $h-(b-c)=d$, or $b-c=h-d$... (1).

We have also $b^2+c^2-2bc\cos A=a^2$, which by means of (1) changes into $4bc\sin^2\frac{1}{2}A=a^2-(h-d)^2$, whence $bc=\dfrac{a^2-(h-d)^2}{4\sin^2\frac{1}{2}a}$; but $bc\sin A=ah$.

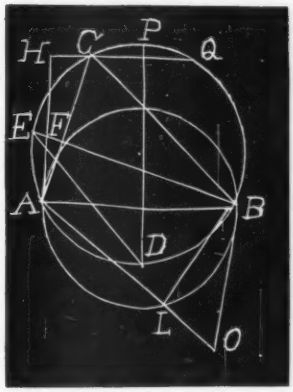
$$\therefore \frac{a^2-(h-d)^2}{4\sin^2\frac{1}{2}a}=\frac{an}{\sin A},$$ and thus we finally get the quadratic equa-

tion in h, $h^2-2(d-a\tan\frac{1}{2}A)h=a^2-d^2$. This quadratic is easily constructed, and after having h the construction of the triangles is very easy.

Solved similarly by V. M. Spunar.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $AB=a$ be the base, ACB the given vertical angle, $p=z-(q-y)$ where z=altitude; x, y the remaining sides, respectively. On AB describe the segment of the circle containing the given vertical angle. Draw the diameter PD perpendicular to AB. Take $DE=AB$ and draw BE. Erect AF perpendicular to AB and also describe a circle on AB as diameter. Draw a line $AL=p$ meeting this circumference in L. Draw LB. Take $LO=AL-AF$ where LO is AL produced. Draw BO. Produce AF to H making $AH=BO+LO$, and draw HCQ parallel to AB. Then ACB is the triangle required. For $xy\sin C=az$, $x-y=z-p$, $a^2=x^2+y^2-2xy\cos C$. Hence $(z-p)^2=a^2-2xy(1-\cos C)=a^2-2az\tan\frac{1}{2}C$.



$\therefore z=p-a\tan\frac{1}{2}C+\sqrt{[(p-a\tan\frac{1}{2}C)^2+a^2-p^2]}$.

$\angle ABE=\frac{1}{2}C$, $AF=a\tan\frac{1}{2}C$, $AL=p$, $BL=\sqrt{[a^2-p^2]}$.

$LO=p-a\tan\frac{1}{2}C$, $BO=\sqrt{[(p-a\tan\frac{1}{2}C)^2=a^2-p^2]}$.

$\therefore AH=z$ and ABC is the required triangle.

CALCULUS.

292. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the partial differential equation, $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=axy$.

Solution by PROF. W. W. BEMAN, University of Michigan.

This is problem 12, page 297, of Johnson's *Differential Equations*.

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{dxy}.$$

From the first two fractions,

$$\frac{1}{x}-\frac{1}{y}=i. \quad \therefore x=\frac{y}{1+cy}.$$

Substituting this value of x in the last two fractions,

$$\frac{dz}{a}=\frac{dy}{1+cy}, \text{ or } z=\frac{a}{c}\log(1+cy)+c'.$$

Replacing c by $\frac{1}{x}-\frac{1}{y}$, $z=\frac{axy}{y-x}\log\frac{y}{x}+\phi\left(\frac{1}{x}-\frac{1}{y}\right)$.

Also solved similarly by J. Scheffer, and G. B. M. Zerr.

293. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find the length of the integral curve of the differential equation $(y^2x^{\frac{1}{3}}+2)dx-x^{\frac{4}{3}}dy=0$ between $x_1=1$ and $x_2=8$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $y=-1/z$, $x^{-\frac{1}{3}}=v$, then the equation becomes $dz+6z^2dv+3v^{-4}dv=0$.

Now, let $z=\frac{1}{6v}+\frac{u}{v^2}$.

Then $v^2du+6u^2dv+3dv=0$, or $\frac{du}{3+6u^2}=-\frac{dv}{v^2}$.

$$\therefore \frac{1}{3\sqrt{2}}\tan^{-1}(u\sqrt{2})=\frac{1}{v}+a=a+x^{\frac{1}{3}} \ldots (1).$$

$$\therefore u=\frac{1}{\sqrt{2}}\tan[3\sqrt{2}(a+x^{\frac{1}{3}})] \ldots (2).$$

$$\therefore y=-\frac{6\sqrt{2}}{6x^{\frac{2}{3}}\tan[3\sqrt{2}(a+x^{\frac{1}{3}})]+x^{\frac{1}{3}}\sqrt{2}} \text{ is the equation.}$$

From (1), $u=\frac{1}{\sqrt{2}}\tan\left[\frac{3\sqrt{2}(av+1)}{v}\right]$.

$$S=\int\sqrt{[1+(du/dv)^2]}dv=\int_{\frac{1}{2}}^{1}\frac{1}{v^2}\sqrt{v^4+9\sec^4\left[\frac{3\sqrt{2}(av+1)}{v}\right]}dv.$$

294. Proposed by C. N. SCHMALL, New York City.

Examine the function, $f(x)=\frac{(x-1)(x-2)}{(x-3)}$ and determine why its *minimum* value is *greater* than its maximum.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

The derivative, $f'(x)=\frac{x^2-6x+7}{(x-3)^2}$, vanishes for $x=3\pm\sqrt{2}$, changing its sign from + to − for the smaller of these values and from − to + at the larger. Thus the function decreases from its maximum value $f(3-\sqrt{2})$ to its minimum $f(3+\sqrt{2})$, *except that it passes through* ∞ at $x=3$, and thus it is possible for the minimum to exceed the maximum. The same fact holds for the more general function of the same type: $F(x)=\frac{(x-a)(x-b)}{x-c}$, if $c>a$ and $c>b$; but the maximum of $F(x)$ exceeds the minimum if $c<a$ and $c<b$. Thus the property proposed for explanation depends not so much on the mere numerical values, nor even on the fact of an infinite discontinuity, as upon the order of the zeros of the numerator and denominator.

Also solved by S. G. Barton, J. E. Sanders, V. M. Spunar, and J. Scheffer.

MECHANICS.

245. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A body moves with constant speed in the circumference of an ellipse. Find the rate of approach (1) to the center, (2) to one of the foci, for any point in the ellipse.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

Differentiating $b^2x^2+a^2y^2=a^2b^2$ with respect to the time, we have $\frac{dx}{dt}/a^2y=-\frac{dy}{dt}/b^2x$, $=-\lambda$ (say), whence if k denote the constant speed, $k^2=\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\lambda^2[a^4y^2+b^4x^2]$, or $\lambda=k/\sqrt{[a^4y^2+b^4x^2]}$. The negative sign should be taken if the motion is clockwise.

Hence, $\frac{dx}{dt}=-ka^2y/\sqrt{[a^4y^2+b^4x^2]}$ and $\frac{dy}{dt}=kb^2x/\sqrt{[a^4y^2+b^4x^2]}$.

(I) Now the radius vector from the center to (x, y) is given by $r^2=x^2+y^2$; whence the rate of approach to the center is

$$-\frac{dr}{dt}=-(x\frac{dx}{dt}+y\frac{dy}{dt})/r=-\frac{kxy(b^2-a^2)}{r\sqrt{[a^4y^2+b^4x^2]}}=\frac{ka^2e^2xy}{\sqrt{[(a^4y^2+b^4x^2)(x^2+y^2)]}}.$$

(II) The radius vector from the left-hand focus $(-ae, 0)$ to (x, y) is given by: $r^2=(x+ae)^2+y^2$; whence

$$-\frac{dr}{dt}=-[(x+ae)\frac{dx}{dt}+y\frac{dy}{dt}]/r=\frac{ka^2ey(ex+a)}{\sqrt{[(a^4y^2+b^4x^2)(x^2+y^2)]}}.$$

REMARK. In (I) the sign changes with either x or y, that is, at the ends of either axis; while in (II) the sign changes only at the ends of the major axis. The factor $ex+a$ vanishes only on the left-hand directrix; hence not in the ellipse.

Also solved by G. B. M. Zerr, and J. Scheffer.

246. Proposed by A. M. HARDING, Adjunct Professor, University of Arkansas, Fayetteville, Ark.

A pentagon $ABCDE$, formed of equal uniform heavy rods connected by smooth joints at their ends, is supported symmetrically in a vertical plane with A uppermost, and AB and AE in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular, the pegs must divide AB and AE each in the ratio $1+\sin(\pi/10):3\sin(\pi/10)$. Jeans' *Theoretical Mechanics*, page 112, number 13.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

Denote by P the point of contact of AB with one peg, and let $AP=2m$ and $PB=2n$. Also let the weight of each rod by $2W$, and the resistance exerted by each peg be R.

Consider first the rod AB, making an angle of 36° with the horizontal. At A the reaction of AE is horizontal (say X_a); for if there were a vertical component, then AB would exert upon AE an oppositely directed vertical component, contrary to the hypothesis of symmetry. At B the reaction of BC is unknown; denote its components by X_b and Y_b. The other forces applied to AB are its weight $2W$, and the reaction R inclined 54° to the horizontal. Equating to zero the sums of the vertical components, horizontal components, and moments about P we obtain (since the distance from P to the midpoint of AB is $2m-(m+n)=m-n$).

$$(1),\ (2) \qquad Y_b+R\sin 54^\circ-2W=0, \quad X_a+X_b+R\cos 54^\circ=0,$$

$$(3) \qquad X_b 2n\cos 54^\circ+Y_b 2n\sin 54^\circ+2W(m-n)\sin 54^\circ-X_a 2m\cos 54^\circ=0.$$

Now R is easily found by considering the pentagon as a whole, to which are applied three external forces, R, R and the weight $10W$. The vanishing of the vertical component gives

$$(4) \qquad 2R\sin 54^\circ-10W=0, \quad R=5W\csc 54^\circ.$$

To obtain X_b consider the rod BC, to which is applied at B the reaction of AB whose components are respectively $(-X_b)$ and $(-Y_b)$. Equating to zero the resultant moment about C, we have, since $BC=2(m+n)$ and is inclined 72° to the horizontal:

$$(5) \qquad (-Y_b).2(m+n)\sin 18^\circ-(-X_b)2(m+n)\cos 18^\circ-2W(m+n)\sin 18^\circ=0.$$

Using (4) to solve successively (1), (5), and (2) we obtain

$$Y_b = -3W, \quad X_b = (W + Y_b)\tan 18^\circ = -2W\tan 18^\circ,$$
$$X_a = 2W\tan 18^\circ - 5W\cot 54^\circ.$$

Putting these values into (3) and dividing by $2W$ we get

$$(6) \qquad -2\tan 18^\circ . n . \cos 54^\circ - 3W . n \sin 54^\circ + (m-n)\sin 54^\circ + (5\cot 54^\circ - 2\tan 18^\circ) . m \cos 54^\circ = 0.$$

Multiplying (6) by $\sin 54^\circ$, collecting m and n, and later employing the identities $2\sin^2 x = 1 - \cos 2x$, $2\cos^2 x = 1 + \cos 2x$, and $2\sin x \cos x = \sin 2x$, we obtain:

$$m[\sin^2 54^\circ - 2\sin 54^\circ \cos 54^\circ \tan 18^\circ + 5\cos^2 54^\circ] = n[4\sin^2 54^\circ + 2\sin 54^\circ \cos 54^\circ \tan 18^\circ],$$
$$m[3 + 2\cos 108^\circ - \sin 108^\circ \tan 18^\circ] = n[2 - 2\cos 108^\circ + \sin 108^\circ \tan 18^\circ],$$
$$\text{or, } m[3 - 3\sin 18^\circ] = n[2 + 3\sin 18^\circ],$$
$$\text{whence, } m : n = 2 + 3\sin(\pi/10) : 3 - 3\sin(\pi/10).$$

The proposed result follows immediately from this one, if we recall from geometry that the half-side of the regular inscribed decagon $= R(\sqrt{5}-1)/4$, so that $\sin 18^\circ$ satisfies the equation

$$(4\sin x + 1)^2 = 5 \text{ or } 4\sin^2 x + 2\sin x = 1.$$

For from this we obtain $\sin x(3\sin x + 2) = 1 - \sin^2 x$, or $2 + 3\sin x : 3(1 - \sin x) :: 1 + \sin x : 3\sin x$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

343. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

A, on contracting to execute a piece of work for $300 and finding after working alone one day that he had finished but 1% of the entire work, engaged B to assist him at the beginning of the second day, with the understanding, that B on each day was to do 6% as much work as had been completed previously, while A each day was to do an amount of

work equal to 1% of the unfinished work at the close of the day before. At the completion of all the work the $300 were divided between A and B in proportion to the amount of the work each had performed.

Required—(1) The number of days to do the work; (2) on which day would the daily earnings of A and B be the same; and (3) the amount of money each was paid under the agreement.

344. Proposed by V. M. SPUNAR, Cleveland, Ohio.

Given $x^7-5x^2y^4=-1506...(1)$, and $y^5-3xy=103...(2)$; find the values of x and y.

GEOMETRY.

374. Proposed by PROF. R. C. ARCHIBALD, Brown University.

The locus of the middle points of chords, of a conic, which all pass through a fixed point P, is a conic. In general, four chords equal to a given length K can be drawn through P. Show that the middle points of these equal chords lie on a circle whose center is independent of K.

375. Proposed by C. N. SCHMALL, New York City.

From a point P on a circle there are drawn three chords PA, PB, PC. Show that the circles described on these chords as diameters intersect again in three collinear points.

376. Proposed by S. LEFSEHETZ, East Pittsburg, Pa.

Inscribe in a given circle a quadrilateral, having given the three diagonals.

CALCULUS.

300. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the differential equation obtaining the complete primitive:
$(x^2+x^2y+2xy-y^2-y^3)dx+(y^2+xy^2+2xy-x^2-x^3)dy=0.$

301. Proposed by C. N. SCHMALL, New York City.

Show that the volume of the surface,
$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$, is $\frac{100\,\pi\,abc}{3.7.11.13}$.

BOOKS AND PERIODICALS.

College Algebra. By Schuyler C. Davidson, Sc. D., Professor of Mathematics in Indiana University. 8vo. Cloth sides and leather back, xiv+243 pp. Price, $1.50 net. New York: The Macmillan Co.

This book, we are told in the preface, is not written for the mathematician but for students wishing to know the elements of ordinary algebra. For this reason, the book is

not exhaustive, many subjects belonging to a treatise on Algebra being omitted. The first chapter deals with the natural number system and the second with rational numbers. The discussion carried on in these two chapters is admirable, but is, in our judgment, unintelligible to the average teacher into whose hands the book is intended to fall. However, much good may be done for this class of teachers by putting before them something lying outside the familiar path of their limited experience.

A sufficient number of problems are given to illustrate the underlying theory. The book will prove serviceable in classes whose previous preparation has been satisfactory. F.

Wentworth's Plane Geometry. Revised by David Eugene Smith. 8vo. Cloth, vi+287 pages. Price, 80 cents. Boston and Chicago: Ginn & Co.

This book is a revision of Wentworth's Plane Geometry, and has many points of superiority over the earlier editions. For example, in the introduction emphasis is laid on the instruments of geometry and practical exercises are given. The book closes with an article on geometric recreations and one on the history of geometry. F.

A Text-Book on Advanced Algebra and Trigonometry With Tables. By William Charles Brenke, Ph. D., Associate Professor of Mathematics in the University of Nebraska. 8vo. Cloth, vii+345 pages. Price, $2.00. New York: The Century Co.

The author of this book believes that the presentation of the two subjects, Algebra and Trigonometry, in a correlated manner, is more satisfactory than to take up the two subjects alternately. For that reason he has written this text. In it are to be found a fairly complete treatment of the ordinary subjects usually studied in the earlier part of a mathematical course. An introduction to the Differential Calculus is also inserted by making use of the derivative. To those teachers who share the author's belief, the book will be found serviceable. F.

Lectures on the Theory of Elliptic Functions. By Harris Hancock, Ph. D. (Berlin), Dr. Sc. (Paris), Professor of Mathematics in the University of Cincinnati. Vol. 1 Analysis. First Edition. First Thousand. Large 8vo. Cloth, xiii+498 pages, 76 figures. Price, $5.00. New York: John Wiley & Sons.

This is by far the most exhaustive and scholarly work that has thus far been published on this subject in America. It is the purpose of the author to present the Theory of Elliptic Functions in three volumes, which are to include the three following phases of the subject, viz: Vol. I, *Analysis*; Vol. II, *Applications to Problems in Geometry and Mechanics*; and Vol. III, *General Arithmetic and Higher Algebra.* In the development of the subject, the author places the Theory of Weierstrass side by side with the Theory of the older writers and many of the formulæ derived by him are contrasted with the corresponding formulæ of the earlier writers. The general theory is treated by means of Riemann surfaces, thus showing the intimate relations between the theory of Weierstrass and his predecessors.

It is to be hoped that the author will soon present the other two volumes to the American mathematicians, who are already greatly indebted to him for this first volume. F.

The Monist for July contains three important mathematical articles. The first is Mathematical Creation, by Henri Poincare; the second, The Construction of Magic Squares and Rectangles by the Method of "Complimentary Differences," by W. S. Andrews; the third, Magic Circles and Spheres, by Harry A. Sayles.

Practical Algebra, First Year Course. By Joseph V. Collins, Professor of Mathematics, State Normal School, Stevens Point, Wisconsin. 8vo. Cloth, 301 pages. Price, $1.00. New York and Chicago: The American Book Co.

This book is an abridgment of the author's Practical Algebra which appeared two years ago and which was noticed in the Monthly at that time. Human interest is added to this text by the insertion of portraits of Newton and Descartes. F.

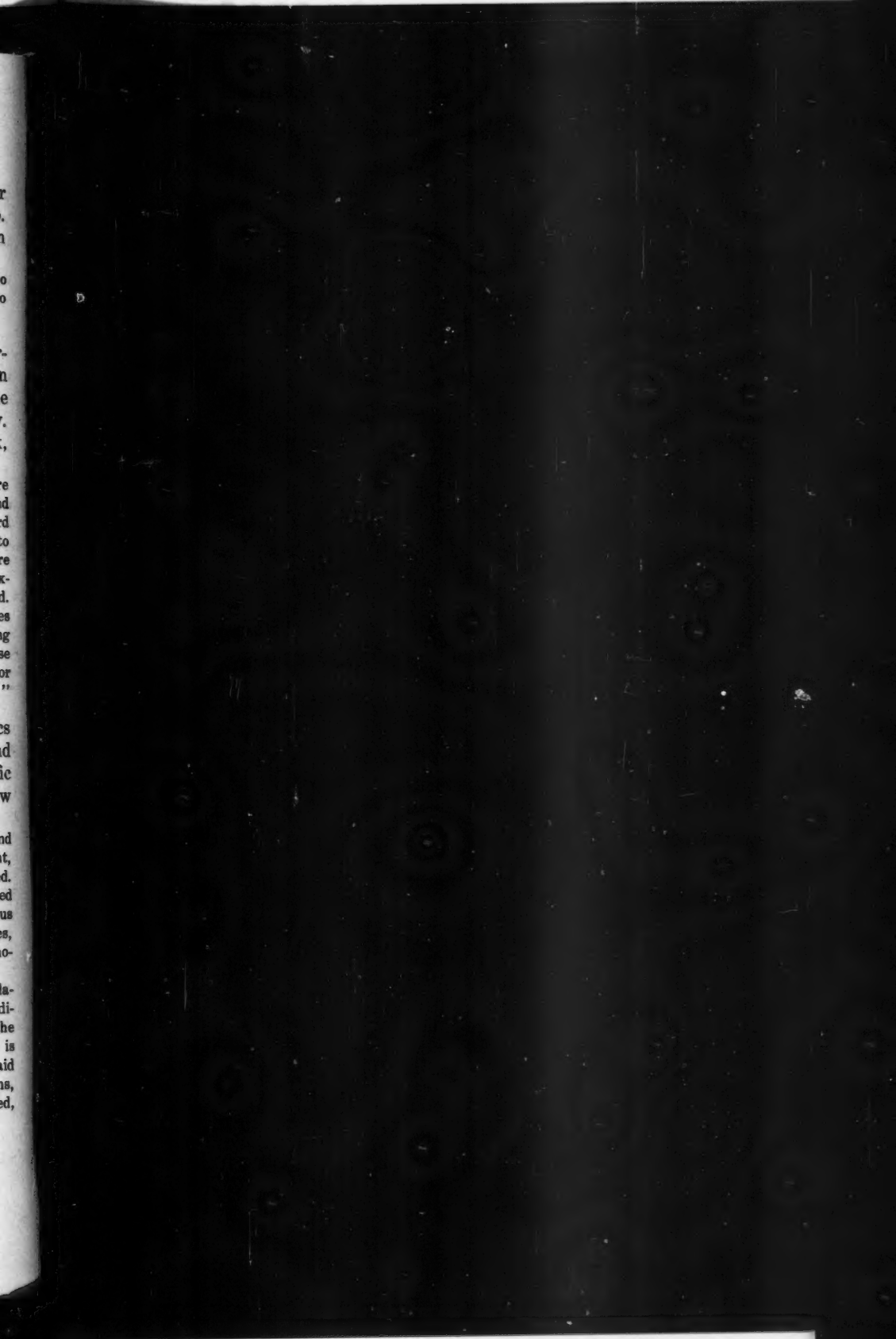
Shop Problems in Mathematics. By William E. Breckenridge, Chairman of the Department of Mathematics, Samuel F. Mersereau, Chairman of the Department of Woodworking, and Charles F. Moore, Chairman of the Department of Metal Working, in Stuyvesant High School, New York City. Cloth, 12mo, 280 pages, illustrated. Price, $1.00. Boston, New York, Chicago: Ginn & Co.

"This book aims to give a thorough training in the mathematical operations that are useful in shop practice, e. g. in Carpentry, Pattern-Making, Foundry Work, Forging, and Machine Work, and, at the same time, to impart to the student much information in regard to shops and shop materials. The mathematical scope varies from addition of fractions to natural trigonometric functions. Problems are graded from simple work in board measure to the more difficult exercises of the machine shop. All problems are based on actual experience. The slide rule is treated at length. Short methods and checks are emphasized. The book should be useful in any schools where there are shops: i. e. in the upper grades of elementary schools for a review course in Practical Mathematics; in Manual Training High Schools as a supplementary book of problems all through the mathematical course and in the shops; in Trade Schools as a text-book either in the mathematics classroom or in the shop; in Normal Schools; in Apprentice Schools, and in the classes of the Y. M. C. A."

Theoretical Mechanics. By Percy F. Smith, Professor of Mathematics in the Sheffield Scientific School, Yale University, and William Raymond Longley, Assistant Professor of Mathematics in the Sheffield Scientific School, Yale University. 8vo. Cloth, 288 pages. List price, $2.50. New York and Chicago: Ginn & Co.

This book is intended for use in courses in mechanics which, as in many colleges and technical schools, are based upon the calculus. For the convenience of the student, formulas from analytic geometry and the calculus, and a table of integrals, are included. The first chapter deals with centers of gravity and moments of inertia. This is followed by chapters on kinematics and kinetics of a particle (including impact), motion in various fields of force (constant field, central field, harmonic field), kinetics of a system of particles, potential, motion in a resisting medium, dynamics of a rigid body, including uniplanar motion, and equilibrium of coplanar forces.

Attention is called to the following special features of the book: 1. The fundamental problem—to determine the motion due to a given force under given initial conditions—is thoroughly discussed. The equations of motion obtained by integration of the force equations have, however, been studied in a previous chapter, and the student is therefore cognizant immediately of the significance of his results. 2. Emphasis is laid everywhere in the solution of problems upon the general application of the force equations, the energy equation, and the impulse equation. 3. The problems are carefully selected, and numerous illustrative examples are worked out in the text.



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ATTEMPTS MADE DURING THE EIGHTEENTH AND NINETEENTH CENTURIES TO REFORM THE TEACHING OF GEOMETRY.*

By FLORIAN CAJORI, Colorado Springs, Colorado.

BIBLIOGRAPHY.

We have found the following six books bearing on the history of the teaching of geometry most useful in making this compilation:

1. V. BOBYNIN — "Elementare Geometrie," being Chapter XXII. in Cantor's *Vorlesungen über Geschichte der Mathematik*, Vol. IV, Leipzig, 1908, pp. 321-402. (Covers second half of eighteenth century.) Referred to as "Bobynin."

2. F. KLEIN — *Elementarmathematik vom Hoeheren Standpunkte aus, Theil II; Geometrie.* Leipzig, 1909, pp. 433-515. Referred to as "Klein."

3. J. PERRY — *Discussion on the Teaching of Mathematics.* British Association Meeting at Glasgow, 1901, London. Referred to as "Perry."

4. H. SCHOTTEN — *Inhalt und Methode des Planimetrischen Unterrichts.* Leipzig, 1890. Referred to as "Schotten."

5. M. SIMON — *Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* Leipzig, 1906. Referred to as "Simon."

6. A. W. STAMPER — *A History of the Teaching of Elementary Geometry*, New York, 1906. Referred to as "Stamper."

Other useful sources of information on the history of the teaching of geometry are as follows:

1. *Reports of the Association for the Improvement of Geometrical Teaching* (in England). The Association now calls itself "The Mathematical Association" and its present organ is the *Mathematical Gazette.*

2. *Zeitschrift für Mathematischen and Naturwissenschaftlichen Unter-*

*This article is a part of the report of the National Committee of Fifteen on a Geometry Syllabus. The Committee has been at work for nearly two years under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences. The Committee is not yet ready to present its report but feels that this historical setting prepared by Professor Cajori should be in the hands of mathematical teachers at once. THE EDITORS.